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A Discrete Periodic Lotka-Volterra System with Delays

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Abstract—In this paper, we investigate the following discrete periodic Lotka-Volterra system with delays:

$$x(n+1) = x(n)\exp\left[b(n) - \sum_{i=1}^{q} a_i(n)\, x(n-\tau_i) - \sum_{j=1}^{m} c_j(n)\, y(n-\ell_j)\right],$$
$$y(n+1) = y(n)\exp\left[r(n) - \sum_{i=1}^{q} d_i(n)\, x(n-k_i) - \sum_{j=1}^{m} e_j(n)\, y(n-s_j)\right].$$

The sufficient and realistic conditions are obtained for the existence of positive periodic solutions and the permanence for the above system. © 2004 Elsevier Ltd. All rights reserved.

Keywords—Periodicity, Discrete Lotka-Volterra system, Positive periodic solutions, Permanence.

1. INTRODUCTION

In this paper, we shall investigate the following periodic:

$$\begin{aligned} x(n+1) &= x(n)\exp\left[b(n) - \sum_{i=1}^{q} a_i(n)\, x(n-\tau_i) - \sum_{j=1}^{m} c_j(n)\, y(n-\ell_j)\right], \\ y(n+1) &= y(n)\exp\left[r(n) - \sum_{i=1}^{q} d_i(n)\, x(n-k_i) - \sum_{j=1}^{m} e_j(n)\, y(n-s_j)\right], \end{aligned} \tag{1.1}$$

where $b(n)$, $r(n)$, $a_i(n)$, $c_j(n)$, $d_i(n)$, and $e_j(n)$ are all positive T-periodic sequences.

We shall consider the solutions of (1.1) with initial condition

$$\begin{aligned} x(-s) &\ge 0, \qquad \text{for } s = 0, 1, \dots, \max_{\substack{i=1,\dots,q \\ j=1,\dots,m}} \{\tau_i, \ell_j\}, \\ y(-s) &\ge 0, \qquad \text{for } s = 0, 1, \dots, \max_{\substack{i=1,\dots,q \\ j=1,\dots,m}} \{k_i, s_j\}. \end{aligned} \tag{1.2}$$

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The purpose of this paper is to investigate the existence of positive periodic solutions and the permanence of the discrete system (1.1).

We say that system (1.1) is permanent if there exists a compact set $D \in \partial\mathbb{R}^2_+$ such that any solution of (1.1) with (1.2) will ultimately stay in D.

In the literature, there is much research about the periodic differential systems, see, for example, [1–4]. But there is very little about the discrete Lotka-Volterra systems, see [5].

In Section 2 of this paper, we shall give a sufficient condition for the existence of positive periodic solutions of (1.1). In Section 3, we shall prove that equation (1.1) is permanent.

2. EXISTENCE OF A POSITIVE PERIODIC SOLUTION

In this section, we shall use Mawhin's continuation theorem of coincidence degree theory to investigate the existence of at least one positive periodic solution of (1.1).

Let $\mathbb{X}$ and $\mathbb{Z}$ be real Banach spaces. Let $\mathbb{L} : \mathbb{Z} \supset \operatorname{dom}\mathbb{L} \to \mathbb{Z}$ be a Fredholm map of index zero and $\mathbb{P} : \mathbb{X} \to \mathbb{X}$, $\mathbb{Q} : \mathbb{Z} \to \mathbb{Z}$ continuous projectors such that $\operatorname{Im}\mathbb{P} = \ker\mathbb{L}$, $\ker\mathbb{Q} = \operatorname{Im}\mathbb{L}$, and $\mathbb{X} = \ker\mathbb{L} \oplus \ker\mathbb{P}$, $\mathbb{Z} = \operatorname{Im}\mathbb{L} \oplus \operatorname{Im}\mathbb{Q}$. Denote by $\mathbb{K}_\mathbb{P} : \operatorname{Im}\mathbb{L} \to \ker\mathbb{P} \cap \operatorname{dom}\mathbb{L}$ the generalized inverse (to $\mathbb{L}$) and by $\mathbb{J} : \operatorname{Im}\mathbb{Q} \to \ker\mathbb{L}$ an isomorphism of $\operatorname{Im}\mathbb{Q}$ on to $\ker\mathbb{L}$.

For the sake of convenience, we introduce Mawhin's continuation theorem as follows.

LEMMA 2.1. *(See [6].) Let Ω be an open bounded set and $\mathbb{N} : \mathbb{X} \to \mathbb{Z}$ be a continuous operator which is $\mathbb{L}$-compact on $\bar{\Omega}$ (i.e., $\mathbb{QN} : \bar{\Omega} \to \mathbb{Z}$ and $\mathbb{K}_\mathbb{P}(\mathbb{I}-\mathbb{P})\mathbb{N} : \bar{\Omega} \to \mathbb{Z}$ are compact). Assume that*

(a) *for each $\lambda \in (0,1)$, every solution x of $\mathbb{L}x = \lambda\mathbb{N}x$ is such that $x \notin \partial\Omega$;*
(b) *$\mathbb{QN}x \neq 0$ for each $x \in \ker\mathbb{Z} \cap \partial\Omega$;*
(c) *the Brouwer degree $\deg(\mathbb{JQN}, \Omega \cap \ker\mathbb{L}, \emptyset) \neq 0$.*

Then the operator equation $\mathbb{L}x = \mathbb{N}x$ has at least one solution in $\operatorname{dom}\mathbb{L} \cap \bar{\Omega}$.

Before we prove our main result, we need the following notations.

NOTATION 1. Set

$$\bar{a}_i = \max_{0\le n\le T-1} a_i(n), \quad \bar{d}_i = \max_{0\le n\le T-1} d_i(n), \quad \bar{c}_j = \max_{0\le n\le T-1} c_j(n), \quad \bar{e}_j = \max_{0\le n\le T-1} e_j(n),$$

$$\underline{a}_i = \min_{0\le n\le T-1} a_i(n), \quad \underline{d}_i = \min_{0\le n\le T-1} d_i(n), \quad \underline{c}_j = \min_{0\le n\le T-1} c_j(n), \quad \underline{e}_j = \min_{0\le n\le T-1} e_j(n),$$

$$\text{for } i = 1,\dots,q \quad \text{and} \quad j = 1,\dots,m,$$

$$\bar{a} = \max_{1\le i\le q}\{\bar{a}_i\}, \quad \bar{d} = \max_{1\le i\le q}\{\bar{d}_i\}, \quad \bar{c} = \max_{1\le j\le m}\{\bar{c}_j\}, \quad \bar{e} = \max_{1\le j\le m}\{\bar{e}_j\},$$

$$\underline{a} = \max_{1\le i\le q}\{\underline{a}_i\}, \quad \underline{d} = \min_{1\le i\le q}\{\underline{d}_i\}, \quad \underline{c} = \min_{1\le j\le m}\{\underline{c}_j\}, \quad \underline{e} = \min_{1\le j\le m}\{\underline{e}_j\},$$

$$\bar{b} = \max_{0\le n\le T-1} b(n), \quad \bar{r} = \max_{0\le n\le T-1} r(n), \quad \underline{b} = \min_{0\le n\le T-1} b(n), \quad \text{and} \quad \underline{r} = \min_{0\le n\le T-1} r(n).$$

NOTATION 2. Set

$$U = \sum_{n=0}^{T-1} e^{u(n)}, \qquad V = \sum_{n=0}^{T-1} e^{v(n)}, \qquad A_i = \sum_{n=0}^{T-1} a_i(n), \qquad D_i = \sum_{n=0}^{T-1} d_i(n),$$

$$C_j = \sum_{n=0}^{T-1} c_j(n), \qquad E_j = \sum_{n=0}^{T-1} e_j(n), \qquad \text{for } i = 1,\dots,q \quad \text{and} \quad j = 1,\dots,m,$$

$$B = \sum_{n=0}^{T-1} b(n) \qquad \text{and} \qquad R = \sum_{n=0}^{T-1} r(n).$$

Next, we need to make the following preparations.

Let

$$\mathbb{X} = \mathbb{Z} = \left\{ z(n) = (u(n), v(n))^T, z(n+T) = z(n) \right\}.$$

Define

$$\|z\| = \max \left\{ \max_{0 \le n \le T-1} |u(n)|, \max_{0 \le n \le T-1} |v(n)| \right\}.$$

Then $(\mathbb{Z}, \|\cdot\|)$ is a Banach space.

Consider an abstract equation in the Banach space $\mathbb{Z}$:

$$\mathbb{L}z = \lambda \mathbb{N} z. \tag{2.1}$$

Define $\mathbb{P}$ and $\mathbb{Q}$ as follows:

$$\mathbb{P}z = \mathbb{Q}z = \begin{pmatrix} \dfrac{1}{T} \sum\limits_{n=0}^{T-1} u(n) \\ \dfrac{1}{T} \sum\limits_{n=0}^{T-1} v(n) \end{pmatrix}.$$

Now, we give the main result of this section.

THEOREM 2.1. *Assume that*

$$\sum_{i=1}^{q} \sum_{j=1}^{m} \left[\bar{a}_i \underline{e}_j - \underline{d}_i \bar{c}_j \right] > 0 \qquad \text{and} \qquad \sum_{j=1}^{m} \left(B \underline{e}_j - R \bar{c}_j \right) > 0, \tag{2.2}$$

$$\sum_{i=1}^{q} \sum_{j=1}^{m} \left[\bar{d}_i \underline{c}_j - \underline{a}_i \bar{e}_j \right] > 0 \qquad \text{and} \qquad \sum_{i=1}^{q} \left(B \bar{d}_i - R \underline{a}_i \right) > 0, \tag{2.3}$$

and

$$\sum_{i=1}^{q} A_i \sum_{j=1}^{m} E_j \ne \sum_{j=1}^{m} C_j \sum_{i=1}^{q} D_i. \tag{2.4}$$

Then equations (1.1) and (1.2) have at least one positive T-periodic solution.

PROOF. It is easy to see that the solutions of (1.1) and (1.2) are positive for $n > 0$. So, we can introduce a change of variables by the following formulas:

$$x(n) = \mathrm{e}^{u(n)} \qquad \text{and} \qquad y(n) = \mathrm{e}^{v(n)}, \tag{2.5}$$

and derive that $u(n)$ and $v(n)$ satisfy the following equations:

$$\begin{aligned} \Delta u(n) &= b(n) - \sum_{i=1}^{q} a_i(n) \mathrm{e}^{u(n-\tau_i)} - \sum_{j=1}^{m} c_j(n) \mathrm{e}^{v(n-\ell_j)}, \\ \Delta v(n) &= r(n) - \sum_{i=1}^{q} d_i(n) \mathrm{e}^{u(n-k_i)} - \sum_{j=1}^{m} e_j(n) \mathrm{e}^{v(n-s_j)}. \end{aligned} \tag{2.6}$$

Set

$$\mathbb{L}z = \Delta z(n) = \begin{pmatrix} \Delta u(n) \\ \Delta v(n) \end{pmatrix} \qquad \text{and} \qquad \mathbb{N}z(n) = \begin{pmatrix} \mathbb{N}u(n) \\ \mathbb{N}v(n) \end{pmatrix},$$

where

$$\mathbb{N}u(n) = b(n) - \sum_{i=1}^{q} a_i(n) \mathrm{e}^{u(n-\tau_i)} - \sum_{j=1}^{m} c_j(n) \mathrm{e}^{v(n-\ell_j)}$$

and

$$\mathbb{N}v(n) = r(n) - \sum_{i=1}^{q} d_i(n)\,\mathrm{e}^{u(n-k_i)} - \sum_{j=1}^{m} e_j(n)\,\mathrm{e}^{v(n-s_j)}.$$

Then $\mathbb{L}$ is a Fredholm mapping of index 0, $\mathbb{N}$ is $\mathbb{L}$-compact on $\bar{\Omega}$ with every Ω open and bounded in $\mathbb{Z}$.

Suppose that $z = z(n) = (u(n), v(n))^T \in \mathbb{Z}$ is a solution of (2.1) for a certain $\lambda \in (0,1)$. Then, we have

$$\begin{aligned}\Delta u(n) &= \lambda\left[b(n) - \sum_{i=1}^{q} a_i(n)\,\mathrm{e}^{u(n-\tau_i)} - \sum_{j=1}^{m} c_j(n)\,\mathrm{e}^{v(n-\ell_j)}\right],\\ \Delta v(n) &= \lambda\left[r(n) - \sum_{i=1}^{q} d_i(n)\,\mathrm{e}^{u(n-k_i)} - \sum_{j=1}^{m} e_j(n)\,\mathrm{e}^{v(n-s_j)}\right].\end{aligned} \tag{2.7}$$

Summing (2.7) over $[0, T-1]$, we get

$$\begin{aligned}\sum_{n=0}^{T-1}\left[b(n) - \sum_{i=1}^{q} a_i(n)\,\mathrm{e}^{u(n-\tau_i)} - \sum_{j=1}^{m} c_j(n)\,\mathrm{e}^{v(n-\ell_j)}\right] &= 0,\\ \sum_{n=0}^{T-1}\left[r(n) - \sum_{i=1}^{q} d_i(n)\,\mathrm{e}^{u(n-k_i)} - \sum_{j=1}^{m} e_j(n)\,\mathrm{e}^{v(n-s_j)}\right] &= 0.\end{aligned} \tag{2.8}$$

From (2.7) and (2.8), we have

$$\sum_{n=0}^{T-1}|\Delta u(n)| \le \sum_{n=0}^{T-1} b(n) + \sum_{n=0}^{T-1}\left[\sum_{i=1}^{q} a_i(n)\,\mathrm{e}^{u(n-\tau_i)} + \sum_{j=1}^{m} c_j(n)\,\mathrm{e}^{v(n-\ell_j)}\right] \le 2\sum_{n=0}^{T-1} b(n) =: M_1,$$

i.e.,

$$\sum_{n=0}^{T-1}|\Delta u(n)| \le M_1. \tag{2.9}$$

Similarly, we have

$$\sum_{n=0}^{T-1}|\Delta v(n)| \le 2\sum_{n=0}^{T-1} r(n) =: M_2. \tag{2.10}$$

It follows from (2.8) that there exist $i_1 \in \{1,\dots,q\}$, $j_1 \in \{1,\dots,m\}$, $n_1, n_2 \in [0, T-1]$, and a constant M_3 which only depend on M_1 such that $u(n_1 - \tau_{i_1}) < M_3$ and $v(n_2 - \ell_{j_1}) < M_3$. Denote $n_1 - \tau_{i_1} = n_1{}' + p_1 T$, $n_2 - \ell_{j_1} = n_2{}' + p_2 T$, $n_1{}', n_2{}' \in [0, T-1]$, where p_1 and p_2 are both integers. Then $u(n_1{}') < M_3$ and $v(n_2{}') < M_3$.

By (2.9) and the above, we have

$$u(n) \le u(n_1{}') + \sum_{n=0}^{T-1}|\Delta u(n)| < M_3 + M_1 =: M_4. \tag{2.11}$$

Similarly, we have

$$v(n) \le v(n_2{}') + \sum_{n=0}^{T-1}|\Delta v(n)| < M_3 + M_2 =: M_5. \tag{2.12}$$

Again, from (2.8), we have

$$\begin{aligned} B &\le U\sum_{i=1}^{q}\overline{a}_i + V\sum_{j=1}^{m}\overline{c}_j, \\ R &\ge U\sum_{i=1}^{q}\underline{d}_i + V\sum_{j=1}^{m}\underline{e}_j \end{aligned} \tag{2.13}$$

and

$$\begin{aligned} B &\ge U\sum_{i=1}^{q}\underline{a}_i + V\sum_{j=1}^{m}\underline{c}_j, \\ R &\le U\sum_{i=1}^{q}\overline{d}_i + V\sum_{j=1}^{m}\overline{e}_j, \end{aligned} \tag{2.14}$$

where B, R, U, V, $\overline{a}_i$, $\overline{c}_j$, $\underline{d}_i$, $\underline{e}_j$, $\underline{a}_i$, $\underline{c}_j$, $\overline{d}_i$, and $\overline{e}_j$ are all defined as those in Notations 1 and 2. Then, from (2.13) and (2.2), we have

$$\sum_{i=1}^{q}\sum_{j=1}^{m}\left(\overline{a}_i\underline{e}_j - \underline{d}_i\overline{c}_j\right)U \ge \sum_{j=1}^{m}\left(B\underline{e}_j - R\overline{c}_j\right) =: M_6 > 0.$$

From (2.14) and (2.3), we have

$$\sum_{i=1}^{q}\sum_{j=1}^{m}\left(\overline{d}_i\underline{c}_j - \underline{a}_i\overline{e}_j\right)V \ge \sum_{i=1}^{q}\left(\overline{d}_iB - \underline{a}_iR\right) =: M_7 > 0.$$

Hence, there exist $i_2 \in \{1,\dots,q\}$, $j_2 \in \{1,\dots,m\}$, $n_3, n_4 \in [0, T-1]$, a constant M_8 which only depends on M_6 and a constant M_9 which only depends on M_7 such that $u(n_3 - \tau_{i_2}) > -M_8$ and $v(n_4 - \ell_{j_2}) > -M_9$. Denote $n_3 - \tau_{i_2} = n_3' + p_3T$, $n_4 - \ell_{j_2} = n_4' + p_4T$, $n_3', n_4' \in [0, T-1]$, where p_3 and p_4 are both integers. Then

$$u\left(n_3'\right) > -M_8 \qquad \text{and} \qquad v\left(n_4'\right) > -M_9. \tag{2.15}$$

By (2.15), (2.9), and (2.10), we have

$$u(n) \ge u\left(n_3'\right) - \sum_{n=0}^{T-1}\left|\Delta u(n)\right| > -\left(M_8 + M_1\right) =: M_{10} \tag{2.16}$$

and

$$v(n) \ge v\left(n_4'\right) - \sum_{n=0}^{T-1}\left|\Delta v(n)\right| > -\left(M_9 + M_2\right) =: M_{11}. \tag{2.17}$$

From (2.11), (2.12), (2.16), and (2.17) we have

$$\|z\| < \max\left\{M_4, M_5, M_{10}, M_{11}\right\} =: M_{12}.$$

Obviously, M_{11} is independent of λ and $z(n)$. From the definition, $\ker\mathbb{L} = \{(h_1, h_2) : h_1$ and h_2 are both constants$\}$. For any $z \in \ker\mathbb{L}$, $\mathbb{QN}z = 0$ is equivalent to the following equations:

$$\begin{aligned} &\sum_{n=0}^{T-1} b(n) - \sum_{n=0}^{T-1}\sum_{i=1}^{q} a_i(n)\,\mathrm{e}^{h_1} - \sum_{n=0}^{T-1}\sum_{j=1}^{m} c_j(n)\,\mathrm{e}^{h_2} = 0, \\ &\sum_{n=0}^{T-1} r(n) - \sum_{n=0}^{T-1}\sum_{i=1}^{q} d_i(n)\,\mathrm{e}^{h_1} - \sum_{n=0}^{T-1}\sum_{j=1}^{m} e_j(n)\,\mathrm{e}^{h_2} = 0. \end{aligned} \tag{2.18}$$

By (2.4), (2.18) has the unique solution (h_1, h_2).

Now, let $M = \max\{M_{12}, |h_1|, |h_2|\}$ and $\Omega = \{z \in \mathbb{Z} : \|z\| < M\}$. It is clear that Ω verifies Requirement (a) of Lemma 2.1. It is easy to show that Ω verifies Requirement (b) of Lemma 2.1 and $\deg\{\mathbb{JQN}, {}^{\prime} \cap \ker\mathbb{L}, \emptyset\} \ne 0$. Hence, (2.6) has at least one T-periodic solution. Through the medium of (2.5), we easily see that (1.1) and (1.2) have at least one positive T-periodic solution. This completes the proof. ∎

3. PERMANENCE

We first establish a lemma which will be used for proving the main result of this section.

LEMMA 3.1. *Any solution $(x(n), y(n))$ of (1.1) and (1.2) is positive and ultimately bounded, i.e.,*

$$\limsup_{n\to\infty} x(n) \le B_1 \qquad \text{and} \qquad \limsup_{n\to\infty} y(n) \le B_2,$$

where

$$B_1 = \max\left\{ \frac{\overline{b}}{\sum_{i=1}^{q} \underline{a}_i} \exp\left\{(\tau+1)\overline{b}\right\}, \frac{1}{\sum_{i=1}^{q} \underline{a}_i} \exp\left\{(\tau+1)\overline{b}-1\right\} \right\},$$

$$B_2 = \max\left\{ \frac{\overline{r}}{\sum_{j=1}^{m} \underline{e}_j} \exp\left\{(s+1)\overline{r}\right\}, \frac{1}{\sum_{j=1}^{m} \underline{e}_j} \exp\left\{(s+1)\overline{r}-1\right\} \right\},$$

$\tau = \max_{1\le i\le q} \tau_i$, *and* $s = \max_{1\le j\le m} s_j$.

PROOF. Clearly, $x(n) > 0$ and $y(n) > 0$ for $n \ge 0$. We first show that $x(n) \le B_1$.

(i) THE CASE $r_1 \le 1/\mathrm{e}$. We claim that for any sufficiently small $\delta > 0$ there exists a large $N_0 > 0$ such that

$$x(N_0) \le \frac{\overline{b}}{\sum_{i=1}^{q} \underline{a}_i}(1+\delta).$$

If it is not the case, we have

$$x(n) > \frac{\overline{b}}{\sum_{i=1}^{q} \underline{a}_i}(1+\delta), \qquad \text{for all large } n.$$

Hence, from (1.1), we obtain, for all large n,

$$\begin{aligned} x(n+1) &\le x(n)\exp\left\{b(n) - \sum_{i=1}^{q} a_i(n)\, x(n-\tau_i)\right\} \le x(n)\exp\left\{\overline{b} - \sum_{i=1}^{q} \underline{a}_i x(n-\tau_i)\right\} \\ &= x(n)\exp\left\{\overline{b}\left(1 - \sum_{i=1}^{q} \frac{\underline{a}_i}{\overline{b}} x(n-\tau_i)\right)\right\} < x(n)\exp\left\{-\overline{b}\delta\right\}. \end{aligned}$$

This implies $x(n) \to 0$ as $n \to \infty$ which is a contradiction to $x(n) > 1+\delta$ for all large n.

Next, we claim that, for any $m > 0$ and $v > 0$,

$$x(m+v) \le (1+\delta)\frac{\overline{b}}{\sum_{i=1}^{q} \underline{a}_i} \exp\left\{v\overline{b}\right\}. \tag{3.1}$$

If $x(m) \le (1+\delta)\overline{b}/\sum_{i=1}^{q} \underline{a}_i$, then, from (1.1), we have

$$\begin{aligned} x(m+v) &\le x(m+v-1)\exp\{b(n)\} \le x(m+v-1)\exp\left\{\overline{b}\right\} \\ &\le x(m)\exp\left\{v\overline{b}\right\} \le \frac{\overline{b}}{\sum_{i=1}^{q} \underline{a}_i}(1+\delta)\exp\left\{v\overline{b}\right\}. \end{aligned}$$

Thus,

$$x(n) \le (1+\delta) \frac{\bar{b}}{\sum_{i=1}^{q} \underline{a}_i} \exp\left\{(\tau+1)\bar{b}\right\}, \qquad \text{for } N_0 < n \le N_0 + \tau + 1.$$

Now, we shall prove that, for all $n > N_0 + \tau + 1$,

$$x(n) \le (1+\delta) \frac{\bar{b}}{\sum_{i=1}^{q} \underline{a}_i} \exp\left\{(\tau+1)\bar{b}\right\}.$$

Otherwise, there exists an $N_1 \ge N_0 + \tau + 1$ such that

$$x(n) \le (1+\delta) \frac{\bar{b}}{\sum_{i=1}^{q} \underline{a}_i} \exp\left\{(\tau+1)\bar{b}\right\}, \qquad \text{for } N_1 \ge n \ge N_0,$$

and

$$x(N_1+1) > (1+\delta) \frac{\bar{b}}{\sum_{i=1}^{q} \underline{a}_i} \exp\left\{(\tau+1)\bar{b}\right\}. \tag{3.2}$$

By (3.1) and (3.2), we have

$$x(N_1 - \tau_i) > (1+\delta) \frac{\bar{b}}{\sum_{i=1}^{q} \underline{a}_i}, \qquad \text{for } i = 1, \dots, q. \tag{3.3}$$

Then, from (1.1) and (3.3), we have

$$\begin{aligned} x(N_1+1) &\le x(N_1) \exp\left\{ b(N_1) - \sum_{i=1}^{q} a_i(N_1)\, x(N_1 - \tau_i) \right\} \\ &\le x(N_1) \exp\left\{-\bar{b}\delta\right\} < (1+\delta) \frac{\bar{b}}{\sum_{i=1}^{q} \underline{a}_i} \exp\left\{(\tau+1)\bar{b}\right\}, \end{aligned}$$

which is a contradiction to (3.2). Therefore, it follows from the arbitrariness of δ that

$$\limsup_{n\to\infty} x(n) \le \frac{\bar{b}}{\sum_{i=1}^{q} \underline{a}_i} \exp\left\{(\tau+1)\bar{b}\right\} \le B_1.$$

(ii) THE CASE $r_1 > 1/e$. By (1.1), we have

$$x(n - \tau_i) \ge x(n) \exp\left\{-\tau_i \bar{b}\right\} \ge x(n) \exp\left\{-\tau \bar{b}\right\}, \qquad \text{for } i = 1, \dots, q, \tag{3.4}$$

when n is large enough. Then, from (1.1) and (3.4), we have

$$x(n+1) \le x(n) \exp\left\{ \bar{b} \left[1 - \frac{\sum_{i=1}^{q} \underline{a}_i}{\bar{b}} \exp\left\{-\tau\bar{b}\right\} x(n) \right] \right\}, \qquad \text{for large } n. \tag{3.5}$$

Now, we define a function: $g(x) = x \exp\{\bar{b}(1 - ax)\}$, where $a = (\bar{b}/\sum_{i=1}^{q} \underline{a}_i) \exp\{-\tau\bar{b}\}$.

It is easy to see that

$$\max_{0<x<\infty} g(x) = \frac{1}{\bar{b}\underline{a}} \exp\left\{\bar{b} - 1\right\}.$$

Then it follows from (3.5) that, for all large n,

$$x(n+1) \le \frac{1}{\bar{b}} \frac{\bar{b}}{\sum_{i=1}^{q} \underline{a}_i} \exp\left\{\tau \bar{b}\right\} \exp\left\{\bar{b} - 1\right\} = \frac{1}{\sum_{i=1}^{q} \underline{a}_i} \exp\left\{(\tau+1)\bar{b} - 1\right\} \le B_1.$$

Obviously, $\lim_{n\to\infty} x(n) \le B_1$. By arguments similar to those above, we can also show that $\lim_{n\to\infty} y(n) \le B_2$. The proof of Lemma 3.1 is complete. ∎

Now, we give the main result of this section.

THEOREM 3.1. *System (1.1) is permanent if*

$$\underline{b}\,\underline{e} - \bar{c}\bar{r} > 0 \quad \text{and} \quad \underline{a}\,\underline{r} - \bar{d}\bar{b} > 0, \tag{3.6}$$

where $\underline{b}$, $\underline{e}$, $\bar{c}$, $\bar{r}$, $\underline{a}$, $\underline{r}$, $\bar{d}$, and $\bar{b}$ are all defined as those in Notation 1.

PROOF. Let $z(n) = (x(n), y(n))$ be any solution of (1.1) and (1.2). Construct two sequences $V_1(n)$ and $V_2(n)$ as follows:

$$\begin{aligned} V_1(n) &= [x(n)]^{\underline{r}} [y(n)]^{-\lambda_1 \underline{b}} \\ &\times \exp\left\{-\underline{r}\bar{a} \sum_{i=1}^{q} \sum_{p=n-\tau_i}^{n-1} x(p) - \underline{r}\bar{c} \sum_{j=1}^{m} \sum_{p=n-\ell_j}^{n-1} y(p) + \underline{r}\bar{c} \sum_{j=1}^{m} \sum_{p=n-s_j}^{n-1} y(p)\right\} \end{aligned} \tag{3.7}$$

and

$$\begin{aligned} V_2(n) &= [x(n)]^{-\lambda_2 \underline{r}} [y(n)]^{\underline{b}} \\ &\times \exp\left\{-\underline{b}\bar{e} \sum_{j=1}^{m} \sum_{p=n-s_j}^{n-1} y(p) - \underline{b}\bar{d} \sum_{i=1}^{q} \sum_{p=n-k_i}^{n-1} x(p) + \underline{b}\bar{d} \sum_{i=1}^{q} \sum_{p=n-\tau_i}^{n-1} x(p)\right\}, \end{aligned} \tag{3.8}$$

where $\lambda_1 = \underline{r}\bar{c}/\underline{b}\,\underline{e}$ and $\lambda_2 = \underline{b}\bar{d}\underline{r}\,\underline{a}$.

For the sake of convenience, we shall use the convention $\sum_{m=n}^{n-1} \bullet = 0$.

For any sufficiently small $\epsilon > 0$, let

$$m_1 = \exp\left\{-\underline{r}\bar{a}B_1' \sum_{i=1}^{q} \tau_i - \underline{r}\bar{c}B_2' \sum_{j=1}^{m} \ell_j\right\}, \qquad m_2 = \exp\left\{-\underline{b}\bar{e}B_2' \sum_{j=1}^{m} s_j - \underline{b}\bar{d}B_1' \sum_{i=1}^{q} k_i\right\},$$

$$M_1 = \exp\left\{\underline{r}\bar{c}B_2' \sum_{j=1}^{m} s_j\right\}, \qquad \text{and} \qquad M_2 = \exp\left\{\underline{b}\bar{d}B_1' \sum_{i=1}^{q} \tau_i\right\},$$

where $B_1' = B_1 + \epsilon$, $B_2' = B_2 + \epsilon$, B_1 and B_2 are both defined as those in Lemma 3.1. Then it follows from (3.7), (3.8), and Lemma 3.1 there exists a large $N > 0$ such that, for $n \ge N$,

$$0 < x(n) < B_1' \quad \text{and} \quad 0 < y(n) < B_2', \tag{3.9}$$

$$m_1 [x(n)]^{\underline{r}} [y(n)]^{-\lambda_1 \underline{b}} \le V_1(n) \le M_1 [x(n)]^{\underline{r}} [y(n)]^{-\lambda_1 \underline{b}} \tag{3.10}$$

and

$$m_2 [x(n)]^{-\lambda_2 \underline{r}} [y(n)]^{\underline{b}} \le V_2(n) \le M_2 [x(n)]^{-\lambda_2 \underline{r}} [y(n)]^{\underline{b}}. \tag{3.11}$$

By (1.1) and calculating $x(n+1)/x(n)$ and $y(n+1)/y(n)$, we have

$$\left[\frac{x(n+1)}{x(n)}\right]^{\underline{r}} \geq \exp\left\{\underline{br} - \underline{r}\bar{a}\sum_{i=1}^{q} x(n-\tau_i) - \underline{r}\bar{c}\sum_{j=1}^{m} y(n-\ell_i)\right\}$$

and

$$\begin{aligned}\left[\frac{y(n+1)}{y(n)}\right]^{-\lambda_1\underline{b}} &\geq \exp\left\{-\lambda_1\underline{b}\bar{r} + \lambda_1\underline{b}\,\underline{d}\sum_{i=1}^{q} x(n-k_i) + \lambda_1\underline{b}\,\underline{e}\sum_{j=1}^{m} y(n-s_i)\right\}\\ &\geq \exp\left\{-\lambda_1\underline{b}\bar{r} + \lambda_1\underline{b}\,\underline{e}\sum_{j=1}^{m} y(n-s_i)\right\}.\end{aligned}$$

From (3.7) and the above, we have

$$\frac{V_1(n+1)}{V_1(n)} \geq \exp\left\{\underline{b}\,\underline{r} - \lambda_1\underline{b}\bar{r} - \underline{r}\bar{a}qx(n)\right\}.$$

Similarly, we have

$$\frac{V_2(n+1)}{V_2(n)} \geq \exp\left\{\underline{b}\,\underline{r} - \lambda_2\bar{b}\underline{r} - \underline{b}\bar{e}my(n)\right\}.$$

Then, by (3.6), we have

$$\underline{b}\,\underline{r} - \lambda_1\underline{b}\bar{r} = \frac{\underline{r}}{\underline{e}}\left(\underline{b}\,\underline{e} - \bar{c}\bar{r}\right) > 0.$$

So, we can select a sufficiently small $\epsilon > 0$ such that

$$\underline{b}\,\underline{r} - \lambda_1\underline{b}\bar{r} - \epsilon > 0.$$

Then, we have

$$V_1(n+1) \geq V_1(n)\exp\{\epsilon\}, \qquad \text{for } 0 < x(n) \leq \frac{\underline{b}\,\underline{r} - \lambda_1\underline{b}\bar{r} - \epsilon}{\underline{r}\bar{a}q} =: h_1. \tag{3.12}$$

Similarly, we have

$$V_2(n+1) \geq V_2(n)\exp\{\epsilon\}, \qquad \text{for } 0 < y(n) \leq \frac{\underline{b}\,\underline{r} - \lambda_2\bar{b}\underline{r} - \epsilon}{\underline{b}\bar{e}m} =: h_2. \tag{3.13}$$

Now, we define a region D as follows: define two curves L_1 and L_2 by

$$L_1 : x^{\underline{r}}y^{-\lambda_1\underline{b}} = \frac{m_1}{M_1}(u_1h_1)^{\underline{r}}\left(B_2{}'\right)^{-\lambda_1\underline{b}}$$

and

$$L_2 : x^{-\lambda_2\underline{r}}y^{\underline{b}} = \frac{m_2}{M_2}\left(B_1{}'\right)^{-\lambda_2\underline{r}}(u_2h_2)^{\underline{b}},$$

where $u_1 = \exp\{\underline{b} - q\bar{a}B_1{}' - m\bar{c}B_2{}'\}$ and $u_2 = \exp\{\underline{r} - q\bar{d}B_1{}' - m\bar{e}B_2{}'\}$.

Obviously, L_1 and L_2 intersect at a unique point in $\partial\mathbb{R}^2_+$. Let D denote the region enclosed by L_1, L_2, $x = B_1{}'$, and $y = B_2{}'$.

In what follows, we shall prove that $z(n)$ eventually enters and remains in D. First, we show that $z(n)$ eventually lies above the curve L_2. Let $N' \geq N+1$. The proof will be divided into two steps.

STEP 1. We claim that there exists an $n_1 \geq N'$ such that $y(n_1) \geq h_2$. Otherwise, from (3.13), we have $V_2(n) \to \infty$ as $n \to \infty$. On the other hand, from (3.9) and (1.1), we have

$$x(n) \to 0, \qquad \text{as } n \to \infty. \tag{3.14}$$

Then, by (3.12), we have $V_1(n) \to \infty$ as $n \to \infty$. By (3.9) and (3.10), we have $y(n) \to 0$ as $n \to \infty$. Then, by (1.1), we have $x(n+1) > x(n)\exp\{\underline{b}/2\}$ when n is large enough. Thus, $x(n) \to \infty$ as $n \to \infty$. This contradicts (3.14).

STEP 2. We claim that for any $p \geq N'$, $z(p+1)$ lies above the curve L_2 and if $y(p) \geq h_2$, then

$$[x(p+1)]^{-\lambda_2 \underline{r}} [y(p+1)]^{\underline{b}} \geq (B_1')^{-\lambda_2 \underline{r}} (u_2 h_2)^{\underline{b}}. \tag{3.15}$$

In fact, if $y(p) \geq h_2$, from (1.1) and (3.9), we have

$$\begin{aligned} y(p+1) &\geq h_2 \exp\left\{\underline{r} - \overline{d} \sum_{i=1}^{q} x(n-k_i) - \overline{e} \sum_{j=1}^{m} y(n-s_j)\right\} \\ &\geq h_2 \exp\left\{\underline{r} - q\overline{d}B_1' - m\overline{e}B_2'\right\} = u_2 h_2. \end{aligned}$$

Then, from (3.9), we know that (3.15) holds, which shows that $z(p+1)$ lies above the curve L_2.

From Step 1, there exists an $n_1 \geq N'$ such that $y(n_1) \geq h_2$. We shall show that $z(n)$ lies above the curve L_2 for all $n \geq n_1$. Otherwise, from Step 2, there exists an $n_3 \geq n_1 + 1$ such that $z(n)$ lies above the curve L_2 for $n_3 \geq n \geq n_1$ and $z(n_3+1)$ lies below the curve L_2, i.e.,

$$[x(n_3+1)]^{-\lambda_2 \underline{r}} [y(n_3+1)]^{\underline{b}} < \frac{m_2}{M_2} (B_1')^{-\lambda_2 \underline{r}} (u_2 h_2)^{\underline{b}}. \tag{3.16}$$

Then we can see from Step 2 that there exists an $n_2 : n_3 > n_2 > n_1$ such that, for $n_3 \geq n \geq n_2$,

$$y(n) < h_2 \tag{3.17}$$

and

$$[x(n_2)]^{-\lambda_2 \underline{r}} [y(n_2)]^{\underline{b}} \geq (B_1')^{-\lambda_2 \underline{r}} (u_2 h_2)^{\underline{b}}. \tag{3.18}$$

Hence, we have, from (3.11), (3.13), (3.17), and (3.18), that

$$\begin{aligned} [x(n_3+1)]^{-\lambda_2 \underline{r}} [y(n_3+1)]^{\underline{b}} &\geq \frac{V_2(n_3+1)}{M_2} > \frac{V_2(n_3)}{M_2} > \cdots > \frac{V_2(n_2)}{M_2} \\ &\geq \frac{m_2}{M_2} [x(n_2)]^{-\lambda_2 \underline{r}} [y(n_2)]^{\underline{b}} \geq \frac{m_2}{M_2} (B_1')^{-\lambda_2 \underline{r}} (u_2 h_2)^{\underline{b}}, \end{aligned}$$

which is a contradiction to (3.16). Therefore, $z(n)$ lies above the curve L_2 for all $n \geq n_1$. Similarly, we can prove that there exists an $n_4 \geq N'$ such that $z(n)$ lies in the right side of the curve L_1 for all $n \geq n_4$. Consequently, $z(n)$ eventually enters and remains in the region D. The proof is complete. ∎

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